On the Hausdorff dimension of continuous functions belonging to Hölder and Besov spaces on fractal d-sets

Abel Carvalho* and António Caetano†

Abstract

The Hausdorff dimension of the graphs of the functions in Hölder and Besov spaces (in this case with integrability $p \geq 1$) on fractal d-sets is studied. Denoting by $s \in (0, 1]$ the smoothness parameter, the sharp upper bound $\min\{d + 1 - s, d/s\}$ is obtained. In particular, when passing from $d \geq s$ to $d < s$ there is a change of behaviour from $d + 1 - s$ to d/s which implies that even highly nonsmooth functions defined on cubes in $\mathbb{R}^n$ have not so rough graphs when restricted to, say, *rarefied* fractals.

MSC 2010: 26A16, 26B35, 28A78, 28A80, 42C40, 46E35.

Keywords: Hausdorff dimension; box counting dimension; fractals; d-sets; continuous functions; Weierstrass function; Hölder spaces; Besov spaces; wavelets.

Acknowledgements: Research partially supported by Fundação para a Ciência e a Tecnologia (Portugal) through Centro de I&D em Matemática e Aplicações (formerly Unidade de Investigação em Matemática e Aplicações) of the University of Aveiro.

1 Introduction

This paper deals with the relationship between dimensions of sets and of the graphs of real continuous functions defined on those sets and having some prescribed smoothness. First studies in this direction are reported in [7, Chapter 10, § 7], where $\min\{d + 1 - s, d/s\}$ is shown to be an upper bound for the Hausdorff dimension of the graphs of Hölder continuous functions with Hölder exponent $s \in (0, 1)$ and defined on compact subsets of $\mathbb{R}^n$ with Hausdorff dimension equal to d.

*Centro I&D Matemática e Aplicações, Universidade de Aveiro, 3810-193 Aveiro, Portugal, abel.carvalho@ua.pt

†Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal, acaetano@ua.pt (corresponding author)

That the above bound is sharp comes out from [7, Chapter 18, § 7]

A corresponding result involving Besov spaces, on cubes on $\mathbb{R}^n$, with smoothness parameter $s \in (0,1]$ and integrability parameter $p \in (0,\infty]$ was established by F. Roueff [11, Theorem 4.8, p. 77], where it turned out that for $p \geq 1$ the sharp upper bound is $n+1-s$. On the other hand, if upper box dimension is used instead, then the complete picture was settled by A. Carvalho [2], with A. Deliu and B. Jawerth [3] as forerunners (though the latter paper contains a mistake noticed by A. Kamont and B. Wolnik [8] as well as, independently, by A. Carvalho [2]).

Our aim here is to study the corresponding problem for Besov spaces when the underlying domains for the functions are allowed to have themselves non-integer dimensions (as was the case in the mentioned results involving Hölder spaces). More precisely, we consider d-sets in $\mathbb{R}^n$ (with $0 < d \leq n$) for our underlying domains and determine the sharp upper bound for the Hausdorff dimension of the graphs of continuous functions defined on such d-sets and belonging to Besov spaces (with integrability parameter $p \geq 1$), with a prescribed smoothness parameter $s \in (0,1]$. As in the case of Hölder spaces, under the assumption $s \leq d$ we obtain the behaviour $d+1-s$, whereas when $s > d$ the correct sharp upper bound is d/s.

One of the qualitative implications of this change of behaviour for small values of d is the following (we illustrate it in the case of Hölder continuous functions):

Given $s \in (0,1]$, $n \in \mathbb{N}$ and a positive integer $d \in (0,n]$, it is possible to find an Hölder continuous functions defined on a cube in $\mathbb{R}^n$ and with Hölder exponent s whose restriction to some d-set has graph with roughness (as measured by the Hausdorff dimension) as close to $d+1-s$ as one wishes. In particular, if we start with an s close to zero, our restriction to a d-set might give us a function with a graph having dimension close to $d+1$. This is also true for $d \in (0,1)$ as long as $d \geq s$, so in such cases the graphs of our functions, even when these are restricted to d-sets with small d, might gain almost one extra unit of roughness when compared with the domain. Consequently we might be near the right endpoint of the interval obtained in Lemma 2.10. However, when d is allowed to become less than s, the dimension of the corresponding graphs cannot overcome d/s, so letting d tend to zero will result in graphs with dimensions also approaching zero. In other words, in such cases we are definitely near the left endpoint of the interval obtained in Lemma 2.10.

So we show that the same type of phenomenon occurs in the setting of Besov spaces. The proof of the upper bound $d+1-s$ is inspired in the deep proof given by Roueff in the case of having n-cubes for domains. On the other hand, the proof of the sharpness owes a lot to the ideas used by Hunt [5], where a combination between randomness and the potential theoretic method for the estimation of Hausdorff

dimensions has been used.

For the sake of completeness and to help having the more complex results in perspective, we revisit also the simpler setting of Hölder spaces and give shorter proofs than in the setting of Besov spaces. For the reader only interested in the result involving Hölder continuous functions, some material can be skipped: Lemma 2.11; everything after Example 2.16 within subsection 2.2; Theorem 3.3 (and its long proof, of course); everything after the first paragraph in the proof of Corollary 3.4. This material has specifically to do with the proof involving Besov spaces.

2 Preliminaries

In this section we give the necessary definitions concerning the dimensions, sets and function spaces to be considered. We also recall some results and establish others that will be needed for the main proofs in the following section. However, we start by listing some notation which applies everywhere in this paper:

The number n is always considered in $\mathbb{N}$. The closed ball in $\mathbb{R}^n$ with center a and radius r is denoted by $B_r(a)$ and a cartesian product of n intervals of equal length is said to be an n-cube. The notation $|\cdot|$ stands either for the Euclidean norm in $\mathbb{R}^n$ or for the sum of coordinates of a multi-index in $\mathbb{N}_0^n$ and λ_n denotes the Lebesgue measure in $\mathbb{R}^n$.

The shorthand diam is used for the diameter of a set and $\mathrm{osc}_I f$ stands for the oscillation of the function f on the set I (that is, the difference $\sup_I f - \inf_I f$), whereas $\Gamma(f)$ denotes the graph of the function f. The usual Schwartz space of functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}(\mathbb{R}^n)$, its dual $\mathcal{S}'(\mathbb{R}^n)$ being the usual space of tempered distributions.

On the relation side, $a \lesssim b$ (or $b \gtrsim a$) applies to nonnegative quantities a and b and means that there exists a positive constant c such that $a \leq cb$, whereas $a \approx b$ means that $a \lesssim b$ and $b \lesssim a$ both hold. On the other hand, $A \subset B$ applies to sets A and B and is the usual inclusion relation (allowing also for the equality of sets). We use $A \hookrightarrow B$ when continuity of the embedding is also meant, for the topologies considered in the sets A and B.

2.1 Dimensions, sets and functions

We start with the dimensions, after recalling some notions related with measures. These definitions and results are taken from [4], to which we refer for details.

Definition 2.1. (a) A *measure* on $\mathbb{R}^n$ is a function $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$, defined over all subsets of $\mathbb{R}^n$, which satisfies the following conditions: (i) $\mu(\emptyset) = 0$; (ii)

$\mu(U_1) \leq \mu(U_2)$ if $U_1 \subset U_2$; (iii) $\mu(\cup_{k\in\mathbb{N}} U_k) \leq \sum_{k\in\mathbb{N}} \mu(U_k)$, with equality in the case when $\{U_k : k \in \mathbb{N}\}$ is a collection of pairwise disjoint Borel sets.

(b) A *mass distribution* on $\mathbb{R}^n$ is a measure μ on $\mathbb{R}^n$ such that $0 < \mu(\mathbb{R}^n) < \infty$.

(c) The *support* of a measure μ is the smallest closed set A such that $\mu(\mathbb{R}^n \setminus A) = 0$.

Definition 2.2. Let $d \geq 0$, $\delta > 0$ and $\emptyset \neq E \subset \mathbb{R}^n$.

(a) We define $\mathcal{H}^d_\delta(E) := \inf\{\sum_{k\in\mathbb{N}} \operatorname{diam}(U_k)^d : \operatorname{diam}(U_k) \leq \delta \text{ and } E \subset \cup_{k\in\mathbb{N}} U_k\}$.

(b) The quantity $\mathcal{H}^d_\delta(E)$ increases when δ decreases. Hence the following definition, of the so-called d-dimensional *Hausdorff measure*, makes sense: $\mathcal{H}^d(E) := \lim_{\delta\to 0^+} \mathcal{H}^d_\delta(E)$. And it is, indeed, a measure according to the preceding definition.

(c) There exists a critical value $d_E \geq 0$ such that $\mathcal{H}^d(E) = \infty$ for $d < d_E$ and $\mathcal{H}^d(E) = 0$ for $d > d_E$. We define the *Hausdorff dimension* of E as $\dim_H E := d_E$.

Remark 2.3. In order to get to the definiton of Hausdorff dimension of the set E we can restrict consideration to sets U_k which are n-cubes, with sides parallel to the axes, of side length 2^{-j}, with $j \in \mathbb{N}$, and centered at points of the type $2^{-j}m$, with $m \in \mathbb{Z}^n$. This can lead to different values for measures, but will produce the same Hausdorff dimension as in the definition above. We shall take advantage of this later on.

We collect in the following remark some properties concerning Hausdorff dimension that we shall also need:

Remark 2.4. (a) $\dim_H \mathbb{R}^n = n$. More generally, the same is true for the dimension of any open subset of $\mathbb{R}^n$.

(b) If $E \subset F$, then $\dim_H E \leq \dim_H F$.

(c) The (Hausdorff) dimension does not increase under a Lipschitzian transformation of sets.

(d) Consider a closed subset E of $\mathbb{R}^n$and a mass distribution μ supported on E. Let $t > 0$ be such that$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x-y|^t} \, d\mu(x)\, d\mu(y) < \infty$. Then $\dim_H E \geq t$.

Definition 2.5. Let E be a non-empty bounded subset of $\mathbb{R}^n$. The *upper box counting dimension* of E is the number

$$\overline{\dim}_B E := \limsup_{j\to\infty} \frac{\log_2 N_j(E)}{j},$$

where $N_j(E)$ stands for the number of n-cubes, with sides parallel to the axes, of side length 2^{-j} and centered at points of the type $2^{-j}m$, with $m \in \mathbb{Z}^n$, which intersect E.

Again, we collect in a remark some properties which will be of use later on:

Remark 2.6. (a) Let E be a non-empty bounded subset of $\mathbb{R}^n$. Then $\dim_H E \leq \overline{\dim}_B E$.

(b) Let $\emptyset \neq E \subset \mathbb{R}^n$, $\emptyset \neq F \subset \mathbb{R}^m$, with E bounded. Then $\dim_H(E \times F) \leq \overline{\dim}_B E + \dim_H F$.

Next we define the sets which we want to consider:

Definition 2.7. Let $0 < d \leq n$. A d-set in $\mathbb{R}^n$ is a (compact) subset K of $\mathbb{R}^n$ which is the support of a mass distribution μ on $\mathbb{R}^n$ satisfying the following condition:

$$\exists c_1, c_2 > 0 : \forall r \in (0,1], \forall x \in K, \ c_1 r^d \leq \mu(B_r(x)) \leq c_2 r^d.$$

Remark 2.8. Here we are not following [4], but rather [6], just with the difference that our d-sets are necessarily compact, because we are assuming that our associated measure μ is actually a mass distribution. Otherwise we can follow [6] and conclude that we can take for μ the restriction to K of the d-dimensional Hausdorff measure and that a d-set has always Hausdorff dimension equal to d.

Remark 2.9. Any d-set K in $\mathbb{R}^n$, with $d \in (0, n]$, intersects $\approx r^d$ cubes of any given regular tessellation of $\mathbb{R}^n$ by cubes of sides parallel to the axes and side length r^{-1}, for any given $r \geq r_0 > 0$, r_0 fixed, with equivalence constants independent of r. For a proof, adapt to our setting the arguments in [10, Lemma 2.1.12].

As we shall be interested, later on, to study dimensions of graphs of functions, we consider here a couple of results involving these special sets. We start with a result which gives already some restrictions for the possible values that the Hausdorff dimension of such sets can have.

Lemma 2.10. *If f is a real function defined on a d-set K, then* $\dim_H \Gamma(f) \in [d, d+1]$.

Proof. (i) We prove first that $\dim_H \Gamma(f) \geq d$. Writing the elements of $\Gamma(f)$ in the form (x,t), with $x \in K \subset \mathbb{R}^n$ and $t = f(x) \in \mathbb{R}$, it is clear that $K = P\Gamma(f)$, where $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection defined by $P(x,t) := x$. As is easily seen, P is a Lipschitzian transformation of sets, therefore, by Remarks 2.8 and 2.4(c), $d = \dim_H K \leq \dim_H \Gamma(f)$.

(d) In order to show that, on the other hand, $\dim_H \Gamma(f) \leq d+1$, just notice that Remarks 2.4(a),(b), 2.6(b), 2.9 and Definition 2.5 allow us to write that

$$\dim_H \Gamma(f) \leq \dim_H K \times \mathbb{R} \leq \overline{\dim}_B K + 1 = d + 1.$$

$\square$

Next we state and prove a *seed* for the so-called *aggregation method* considered in [11, p. 24, discussion after Remark 2.2]:

Lemma 2.11. *Let $k, l \in \mathbb{N}_0$, with $k < l$, and h_0, h_1 be two bounded real functions defined on a bounded subset K of $\mathbb{R}^n$. Let $\mathcal{Q}_j$, with $j = k, l$, be finite coverings of K by n-cubes Q_j with sides parallel to the axes, of side length 2^{-j} and centered at points of the type $2^{-j}m$, with $m \in \mathbb{Z}^n$. Assume that a covering of $\Gamma(h_0)$ by $(n+1)$-cubes of side length at least 2^{-k} is given, and such that* over *each Q_k each point between the* levels *$m_{Q_k} := \inf_{Q_k \cap K} h_0$ and $M_{Q_k} := \sup_{Q_k \cap K} h_0$ belongs to one of those $(n+1)$-cubes. Then the number of $(n+1)$-cubes of side length 2^{-l} that one needs to add to the given covering of $\Gamma(h_0)$, in order to get a covering of $\Gamma(h_0 + h_1)$ by $(n+1)$-cubes of side length at least 2^{-l}, and such that* over *each Q_l each point between the* levels *$m_{Q_l} := \inf_{Q_l \cap K} h_0 + h_1$ and $M_{Q_l} := \sup_{Q_l \cap K} h_0 + h_1$ belongs to one of those $(n+1)$-cubes, is bounded above by*

$$\sum_{Q_l \in \mathcal{Q}_l} (2^{l+1} \sup_{y \in Q_l} |h_1(y)| + 2).$$

Proof. Clearly what one needs is to cover the portion of $\Gamma(h_0+h_1)$ *over* each Q_l and then put all together. Notice that, given one such Q_l, what one really needs to cover is the portion of $\Gamma(h_0+h_1)$ *over* $Q_k \cap Q_l$, for the $Q_k$containing Q_l. By hypothesis, all the points of $\mathbb{R}^{n+1}$*over* $Q_k \cap Q_l$ between the *levels* m_{Q_k} and M_{Q_k} are already covered, so one only needs to ascertain which points of $\Gamma(h_0+h_1)$ over $Q_k \cap Q_l$ ($= Q_l$) do not fall within those *levels*. Since

$$m_{Q_k} - \sup_{y \in Q_l}(-h_1(y)) = m_{Q_k} + \inf_{y \in Q_l} h_1(y) \leq h_0(x) + h_1(x) \leq M_{Q_k} + \sup_{y \in Q_l} h_1(y)$$

for all $x \in Q_l \cap K$, then it is enough to add, *over* Q_l, to the previous cover, a number of $(n+1)$-cubes of side length 2^{-l} in a quantity not exceeding $2\,(2^l \sup_{y \in Q_l} |h_1(y)| + 1)$. $\square$

We shall also need a couple of technical lemmas which we state and prove next:

Lemma 2.12. *Let $\beta > \alpha > 0$ be two fixed numbers and $\zeta : (0, \beta) \to \mathbb{R}^+$ be a fixed integrable function. Then*

$$\int_0^\alpha \zeta(r) f(r)\, dr \approx \int_0^\beta \zeta(r) f(r)\, dr$$

for all non-increasing functions $f : (0, \beta) \to \mathbb{R}^+$.

Proof. $\int_\alpha^\beta \zeta(r) f(r)\, dr \leq \int_\alpha^\beta \zeta(r) f(\alpha)\, dr = c \int_0^\alpha \zeta(r) f(\alpha)\, dr \leq \int_0^\alpha \zeta(r) f(r)\, dr.$ $\square$

Lemma 2.13. *Consider $0 < d \leq n$ and K a d-set. Assume that μ_K is a mass distribution supported on K according to Definition 2.7. Then*

$$\int_{\mathbb{R}^n} f(|x-y|)\, d\mu_K(x) \approx \int_0^{\operatorname{diam} K} r^{d-1} f(r)\, dr$$

for all $y \in K$ and all non-increasing and continuous functions $f : \mathbb{R}^+ \to \mathbb{R}^+$.

Proof. Let $R \in (0,\infty)$ be such that $K \subset B_R(y)$, with R chosen independently of y (this is possible, due to the boundedness of K). By the definition of d-set, there exists $c_1, c_2 > 0$ (also independent of y) such that

$$c_1 r^d \leq \mu_K(B_r(y)) \leq c_2 r^d, \qquad \text{for all } r \in (0, R].$$

For any $\lambda \in (0, R)$, consider the annulus

$$C^{(\lambda)} := \{x \in \mathbb{R}^n : \; R - \lambda < |x-y| \leq R\}$$

and, for any $N \in \mathbb{N}$, the partition $\{C_l : \; l = 1, \dots, N\}$ of this annulus, where

$$C_l := \{x \in \mathbb{R}^n : \; R - \frac{l\lambda}{N} < |x-y| \leq R - \frac{(l-1)\lambda}{N}\}, \qquad \text{for } l = 1, \dots, N.$$

We have the following estimations:

$$\begin{aligned}
\int_{C^{(\lambda)}} f(|x-y|)\, d\mu_K(x) &\leq \sum_{l=1}^{N} \mu_K(C_l) f(R - \frac{l\lambda}{N}) \\
&= \varepsilon_N + \sum_{l=1}^{N} \mu_K(C_l) f(R - \frac{(l-1)\lambda}{N}) \\
&\leq \varepsilon_N + \sum_{l=1}^{N} (c_2 r_{l-1}^d - c_2 r_l^d) f(r_{l-1}) \\
&\leq \varepsilon_N + \int_{(c_1/c_2)^{1/d}(R-\lambda)}^{R} \frac{d(c_2 r^d)}{dr} f(r)\, dr \\
&= \varepsilon_N + c_2 d \int_{(c_1/c_2)^{1/d}(R-\lambda)}^{R} r^{d-1} f(r)\, dr
\end{aligned}$$

where $r_l := c_2^{-1/d} \mu_K(B_{R-l\lambda/N}(y))^{1/d}$, so $r_l \leq R - \frac{l\lambda}{N}$, $l = 0, \dots, N$. Furthermore, ε_N equals

$$\begin{aligned}
&\sum_{l=1}^{N} \mu_K(C_l)(f(R - \frac{l\lambda}{N}) - f(R - \frac{(l-1)\lambda}{N})) \\
&\quad \leq \; \mu_K(C^{(\lambda)}) \max_{l=1,\dots,N} |f(R - \frac{l\lambda}{N}) - f(R - \frac{(l-1)\lambda}{N})|,
\end{aligned}$$

from which follows (using the uniform continuity of f in $[R-\lambda, R]$) that $\lim_{N\to\infty} \varepsilon_N = 0$. In this way we obtain

$$\int_{C^{(\lambda)}} f(|x-y|)\, d\mu_K(x) \leq c_2 d \int_{(c_1/c_2)^{1/d}(R-\lambda)}^{R} r^{d-1} f(r)\, dr. \tag{1}$$

And, by similar calculations,

$$\int_{C^{(\lambda)}} f(|x-y|)\, d\mu_K(x) \geq c_1 d \int_{(c_2/c_1)^{1/d}(R-\lambda)}^{R} r^{d-1} f(r)\, dr, \tag{2}$$

as long as we only consider values of $\lambda \in (0,R)$ close enough to R, so that $(c_2/c_1)^{1/d}(R-\lambda) < R$. Letting now λ tend to R in (1) and (2), we get (using also the fact that $\mu_K(\{0\}) = 0$)

$$c_1 d \int_0^R r^{d-1} f(r)\, dr \le \int_{\mathbb{R}^n} f(|x-y|)\, d\mu_K(x) \le c_2 d \int_0^R r^{d-1} f(r)\, dr.$$

The required result now follows by applying Lemma 2.12. $\square$

Example 2.14. As an interesting application of the preceding result, which will be useful to us later on, we have

$$\int_K \frac{1}{|x-y|^u}\, d\mu_K(x) \approx \int_0^{\operatorname{diam} K} r^{d-u-1}\, dr,$$

with equivalence constants independent of $y \in K$ and $u \ge 0$.

2.2 Function spaces

Definition 2.15. For $s \in (0,1]$ and $\emptyset \ne K \subset \mathbb{R}^n$, define $\mathcal{C}^s(K) := \{f : K \to \mathbb{R} : \exists c > 0 : \forall x, y \in K,\ |f(x) - f(y)| \le c\,|x-y|^s\}$. In particular, all functions in $\mathcal{C}^s(K)$ are continuous (of course, considering in K the metric inherited from the sorrounding $\mathbb{R}^n$). We shall call $\mathcal{C}^s(K)$ the set of the (real) Hölder continuous functions (over K) of exponent s. On the other hand, given $r \in \mathbb{N}$, we denote by $C^r(\mathbb{R}^n)$ the set of all complex-valued functions defined on $\mathbb{R}^n$ such that the function itself and all its derivatives up to (and including) the order r are bounded and uniformly continuous.

The following is a non-trivial example of a function in $\mathcal{C}^s(K)$. Its proof follows from an easy adaptation of a corresponding result in [5, p. 796].

Example 2.16. $x \mapsto W_{s,\theta}(x) := \sum_{i=1}^n \sum_{j=0}^\infty \rho^{-js} \cos(\rho^j x_i + \theta_{ij})$ is Hölder continuous of exponent $s \in (0,1)$ on any bounded subset of $\mathbb{R}^n$, with Hölder constant independent of θ.

The following definition (of Daubechies wavelets) includes an existence assertion. For details, we refer to [15, section 3.1].

Definition 2.17. Let $r \in \mathbb{N}$. Define $L_0 := 1$ and $L := L_j := 2^n - 1$ if $j \in \mathbb{N}$. There exist compactly supported real functions $\psi_0 \in C^r(\mathbb{R}^n)$ and $\psi^l \in C^r(\mathbb{R}^n)$, $l = 1, \dots, L$

$$(\text{ with } \int_{\mathbb{R}^n} x^\alpha \psi^l(x)\, dx = 0,\ \alpha \in \mathbb{N}_0^n,\ |\alpha| \le r\ ), \tag{3}$$

such that $\{\Psi^l_{jm} : j \in \mathbb{N}_0,\ 1 \le l \le L_j,\ m \in \mathbb{Z}^n\}$ is an orthonormal basis in $L_2(\mathbb{R}^n)$, where, by definition,

$$\Psi^l_{jm}(x) := \begin{cases} \psi_0(x-m) & \text{if } j = 0,\ l = 1,\ m \in \mathbb{Z}^n \\ 2^{\frac{j-1}{2}n} \psi^l(2^{j-1}x - m) & \text{if } j \in \mathbb{N},\ 1 \le l \le L,\ m \in \mathbb{Z}^n \end{cases}.$$

The Besov spaces in the following definition are the usual ones (up to equivalent quasi-norms), defined by Fourier-analytical tools (a definition along this line can be seen in [15, section 1.3], for example). From this point of view the definition which follows is actually a theorem: for details, see [15, Theorem 3.5 and footnote in p. 156].

Definition 2.18. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and r be a natural number such that $r > \max\{s, n(1/p-1)_+ - s\}$. The Besov space $B^s_{pq}(\mathbb{R}^n)$ is the set of all sums

$$f := \sum_{j,l,m} \lambda^l_{jm} 2^{-jn/2} \Psi^l_{jm} = \sum_{j=0}^{\infty} \sum_{l=1}^{L_j} \sum_{m\in\mathbb{Z}^n} \lambda^l_{jm} 2^{-jn/2} \Psi^l_{jm} \tag{4}$$

(convergence — actually, unconditional convergence — in $\mathcal{S}'(\mathbb{R}^n)$), for all given sequences $\{\lambda^l_{jm} \in \mathbb{C} :\, j \in \mathbb{N}_0,\, l = 1, \ldots, L_j,\, m \in \mathbb{Z}^n\}$ such that

$$\left(\sum_{m\in\mathbb{Z}^n} |\lambda^1_{0m}|^p\right)^{1/p} + \sum_{l=1}^{L} \left(\sum_{j=1}^{\infty} 2^{j(s-n/p)q} \left(\sum_{m\in\mathbb{Z}^n} |\lambda^l_{jm}|^p\right)^{q/p}\right)^{1/q} \tag{5}$$

(with the usual modifications if $p = \infty$ or $q = \infty$) is finite. It turns out that (5) defines a quasi-norm in $B^s_{pq}(\mathbb{R}^n)$ which makes this a complete space.

Remark 2.19. (a) The representation (4) is unique, that is, it is uniquely determined by the limit f, namely the coefficients are determined by the formulæ

$$\lambda^l_{jm} = 2^{jn/2}(f, \Psi^l_{jm}), \quad j \in \mathbb{N}_0,\, l = 1, \ldots, L_j,\, m \in \mathbb{Z}^n,$$

where $(\cdot,\cdot)$, though standing for the inner product in $L_2(\mathbb{R}^n)$ when applied to functions in such a space, must in general be understood in the sense of the dual pairing $\mathcal{S}(\mathbb{R}^n) - \mathcal{S}'(\mathbb{R}^n)$ — see [15, section 3.1] for details. As a consequence, when f is compactly supported, then, given any $j \in \mathbb{N}_0$, only finitely many coefficients λ^l_{jm} are non-zero.

(b) Arguing as in [11, p. 21], we can say that the convergence in (4) is even uniform in the support of f whenever this support is compact and f is a continuous function.

Definition 2.20. Consider $0 < d \leq n$ and K a d-set with associated mass distribution μ according to Definition 2.7. For $0 < p < \infty$, we define the Lebesgue space $L_p(K)$ as the set of all μ-measurable functions $f : K \to \mathbb{C}$ for which the quasi-norm given by

$$\|f\|_{L_p(K)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, d\mu(x)\right)^{1/p}$$

is finite.

Definition 2.21. Consider $0 < d \leq n$ and K a d-set. Let $0 < p, q < \infty$. Assuming that there exists $c > 0$ such that

$$\|\varphi|_K\|_{L_p(K)} \leq c \, \|\varphi\|_{B^s_{pq}(\mathbb{R}^n)}, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n), \tag{6}$$

the trace of $f \in B^s_{pq}(\mathbb{R}^n)$ on K is defined by $tr_K\, f \ := \lim_{j\to\infty} \varphi_j|_K$ in $L_p(K)$, where $(\varphi_j)_j \subset \mathcal{S}(\mathbb{R}^n)$ is any sequence converging to f in $B^s_{pq}(\mathbb{R}^n)$.

This definition is justified by the completeness of $L_p(K)$ and by the fact that the restrictions on p, q guarantee that the Schwartz space is dense in the Besov spaces under consideration. That the definition does not depend on the particular approaching sequence $(\varphi_j)_j$ is a consequence of (6).

By [14, Theorem 18.6 and Comment 18.7], which holds for $d = n$ too (cf. also [1, Theorem 3.3.1(i)]), one knows that the assumption (6) holds true when $0 < p < \infty$, $0 < q \leq \min\{1, p\}$ and $s = \frac{n-d}{p}$. Therefore the trace of functions of Besov spaces on K is well-defined for that range of parameters.

Since $B^s_{pq}(\mathbb{R}^n) \hookrightarrow B^{\frac{n-d}{p}}_{p,\min 1,p}(\mathbb{R}^n)$ whenever $s > \frac{n-d}{p}$, then the trace as defined above makes sense for functions of the spaces $B^s_{pq}(\mathbb{R}^n)$, for any $0 < p < \infty$, $0 < q < \infty$ and $s > \frac{n-d}{p}$. Moreover, since the embedding between the Besov spaces above also hold when $q = \infty$, then, though Definition (2.21) can no longer be applied, we define, for any $f \in B^s_{p\infty}(\mathbb{R}^n)$, with $0 < p < \infty$ and $s > \frac{n-d}{p}$, the $tr_K\, f$ by its trace when f is viewed as an element of $B^{\frac{n-d}{p}}_{p,\min 1,p}(\mathbb{R}^n)$.

Finally, in the case $f \in B^s_{\infty q}(\mathbb{R}^n)$, with $0 < q \leq \infty$ and $s > 0$, we define $tr_K\, f \ := f|_K$, the pointwise restriction, since, for such range of parameters, the elements of those Besov spaces are all (represented by) continuous functions (actually, we even have $B^s_{\infty q}(\mathbb{R}^n) \hookrightarrow C(\mathbb{R}^n)$, where the latter space stands for the set of all complex-valued, bounded and uniformly continuous functions on $\mathbb{R}^n$ endowed with the sup norm).

Remark 2.22. Another way of defining trace on K is by starting to define it by pointwise restriction when the function is continuous and, at least in the case when f is locally integrable on $\mathbb{R}^n$, define its trace on K by the pointwise restriction $\overline{f}|_K$, where

$$\overline{f}(x) \ := \lim_{r\to 0} \frac{1}{\lambda_n(B_r(x))} \int_{B_r(x)} f(y)\, dy$$

for the values of x where the limit exists. It is known (Lebesgue differentiation theorem) that $\overline{f} = f$ a.e. (for locally integrable functions f). As is easily seen, the identity $\overline{f}(x) = f(\mathrm{x})$ surely holds at any point x where f is continuous.

This is the approach followed by Jonsson and Wallin in [6] (see pp. 14-15), where they have shown (it is a particular case of [6, Theorem 2 in p. 142]) that, for $0 <$

$d \leq n$, $1 \leq p, q \leq \infty$ and $s > \frac{n-d}{p}$, the map $f \mapsto \overline{f}|_K$ takes $B^s_{pq}(\mathbb{R}^n)$ linearly and boundedly into $L_p(K)$.

Note now that, for this restriction of parameters, both $tr_K\, f$ and $\overline{f}|_K$ coincide with $f|_K$ when $f \in \mathcal{S}(\mathbb{R}^n)$. Therefore, at least when we further restrict p and q to be finite, we get the identity $tr_K\, f = \overline{f}|_K$ in $L_p(K)$ for any $f \in B^s_{pq}(\mathbb{R}^n)$, by a density argument. In the case we still restrict p to be finite but admit $q = \infty$, from our definition above we see that $tr_K\, f$ is also the trace of $f \in B^{\varepsilon+\frac{n-d}{p}}_{p,\min 1,p}(\mathbb{R}^n)$, for any suitable small $\varepsilon > 0$, where here the parameter "q" is again finite, so also $tr_K\, f = \overline{f}|_K$ in $L_p(K)$. This identity even holds when $p = \infty$ is admitted, taking into account that in that situation we are dealing with continuous functions.

Summing up, when $0 < d \leq n$, $1 \leq p, q \leq \infty$ and $s > \frac{n-d}{p}$ we have $tr_K\, f = \overline{f}|_K$.

Although in [6] both p and q are assumed to be greater than or equal to 1, we can proceed with our comparative analysis between $tr_K\, f$ and $\overline{f}|_K$ even for the remaining positive values of q. In fact, given $0 < d \leq n$, $1 \leq p \leq \infty$, $0 < q < 1$ and $s > \frac{n-d}{p}$, and due to the embedding $B^s_{pq}(\mathbb{R}^n) \hookrightarrow B^{\varepsilon+\frac{n-d}{p}}_{p,\min 1,p}(\mathbb{R}^n)$ (which holds for any suitable small $\varepsilon > 0$), we see, from what was mentioned before, that the trace of $f \in B^s_{pq}(\mathbb{R}^n)$ can be seen as the trace of f as an element of $B^{\varepsilon+\frac{n-d}{p}}_{p,\min 1,p}(\mathbb{R}^n)$; since in the latter space the "q" parameter is in the range $[1, \infty]$ (actually, it is 1), then we already know that $tr_K\, f = \overline{f}|_K$ here too.

As a consequence we have also the following remark, which will be useful later on:

Remark 2.23. Consider $0 < d \leq n$ and K a d-set. If f is a continuous function belonging to $B^s_{pq}(\mathbb{R}^n)$, with $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $s > \frac{n-d}{p}$, then $tr_K\, f = f|_K$.

Definition 2.24. Consider $0 < d \leq n$ and K a d-set. Let $0 < p, q \leq \infty$ and $s > 0$. We define the Besov space $\mathbb{B}^s_{pq}(K)$ as the set of traces of the elements of $B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ endowed with the quasi-norm defined by

$$\|f\|_{\mathbb{B}^s_{pq}(K)} := \inf \|g\|_{B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)}$$

where the infimum runs over all $g \in B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ such that $tr_K\, g = f$.

For a motivation for such definition, see [13, sections 20.2 and 20.3]. From the considerations above it follows that the *trace* maps $B^{s+\frac{n-d}{p}}_{pq}(\mathbb{R}^n)$ linearly and boundedly both into $L_p(K)$ and $\mathbb{B}^s_{pq}(K)$, where the parameters are as in the preceding definition.

Proposition 2.25. *Consider* $0 < d \leq n$ *and* K *a* d*-set. Let* $0 < p_2 < p_1 \leq \infty$*,* $0 < q \leq \infty$ *and* $s > 0$*. Then*

$$\mathbb{B}^s_{p_1 q}(K) \hookrightarrow \mathbb{B}^s_{p_2 q}(K).$$

A sketch of a proof for this result, at least for $d = n$, can be seen in [13, Step 2 in p. 165]. The argument is not clear to us when $0 < d < n$, but a proof in this situation can be seen in [9, Proposition 2.18], where quarkonial decompositions were used. In both cases, a proof with atomic decompositions can also be used instead.

From Remark 2.23 it follows that the trace on a d-set K, with $0 < d \leq n$, of a continuous function belonging to $B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$, with $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $s > 0$, is still a continuous function (on K). In the sequel we shall need a partial converse for this result, the proof of which is sketched below:

Proposition 2.26. *Consider* $0 < d \leq n$ *and* K *a* d*-set. Let* $1 \leq p, q \leq \infty$ *and* $0 < s < 1$*. Any continuous function in* $\mathbb{B}_{pq}^s(K)$ *can be obtained as the trace (or pointwise restriction) of a continuous function in* $B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$*.*

Proof. (i) We start with the case $0 < d < n$.

We shall need to consider a Whitney decomposition of K^c by a family of closed n-cubes Q_i and an associated partition of unity by functions ϕ_i. We use here the notations and conventions of [6, pp. 23-24 and 155-157], except that our K here is in the place of F over there. In particular, x_i, s_i and l_i shall, respectively, stand for the center of Q_i, its side length and its diameter. Moreover, given $f \in \mathbb{B}_{pq}^s(K)$, the function $\mathcal{E}f$ is defined by

$$\mathcal{E}f(x) := \sum_{i \in I} \phi_i(x) \frac{1}{\mu_K(B_{6l_i}(x_i))} \int_{|t-x_i| \leq 6l_i} f(t)\, d\mu_K(t), \qquad x \in K^c,$$

where I is the set of indices i such that $s_i \leq 1$ and μ_K is the mass distribution supported on K according to Definition 2.7. Notice that $\mathcal{E}f$ is defined a.e. in $\mathbb{R}^n$, because the asumption $d < n$ guarantees that K has Lebesgue measure 0.

According to [6, Theorem 3 in p. 155], $\mathcal{E}f \in B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$, $\mathcal{E}f$ is C^∞ in K^c (so, in particular, it is continuous on K^c) and $tr_K \mathcal{E}f = (\overline{\mathcal{E}f})|_K = f$. Since $\overline{\mathcal{E}f} = \mathcal{E}f$ a.e., $\overline{\mathcal{E}f}$ is a representative of an element of $B_{pq}^{s+\frac{n-d}{p}}(\mathbb{R}^n)$ whose pointwise restriction to K coincides with f. Since the identity $\overline{\mathcal{E}f}(x) = \mathcal{E}f(x)$ holds for any $x \in K^c$, where we already know this function is continuous, it remains to show that $\overline{\mathcal{E}f}$ is continuous on K, i.e., that

$$\forall t_0 \in K,\, \forall \varepsilon > 0,\, \exists \delta > 0 : \forall x \in \mathbb{R}^n,\, x \in B_\delta(t_0) \Rightarrow |\overline{\mathcal{E}f}(x) - f(t_0)| < \varepsilon.$$

The implication being trivially true when x is also in K, we can assume that $x \in B_\delta(t_0) \cap K^c$, in which case we have to arrive to the conclusion that $|\mathcal{E}f(x) - f(t_0)| < \varepsilon$.

Pick one Q_k containing x and consider $t_k \in K$ and $y_k \in Q_k$ such that $|y_k - t_k| = \operatorname{dist}\{Q_k, K\}$. Notice that $|t_k - t_0| \leq 3\delta$ and that

$$|\mathcal{E}f(x) - f(t_0)| \leq |\mathcal{E}f(x) - f(t_k)| + |f(t_k) - f(t_0)|,$$

hence, by the continuity of f on K, for sufficiently small $\delta > 0$ one gets $|f(t_k) - f(t_0)| < \varepsilon/2$ and, therefore, we only need to show that $|\mathcal{E}f(x) - f(t_k)| < \varepsilon/2$ too.

Assume that we will choose $\delta > 0$ small enough, so that, in particular, the restriction $s_i \leq 1$ for the indices $i \in I$ is not really a restriction and, therefore, $\sum_{i\in I} \phi_i(x) = 1$. Then

$$
\begin{aligned}
|\mathcal{E}f(x) - f(t_k)| &= |\sum_{i\in I} \phi_i(x) \frac{1}{\mu_K(B_{6l_i}(x_i))} \int_{|t-x_i|\leq 6l_i} f(t)\, d\mu_K(t) - \sum_{i\in I} \phi_i(x) f(t_k)| \\
&\leq \sum_{i\in I} \phi_i(x) \frac{1}{\mu_K(B_{6l_i}(x_i))} \int_{|t-x_i|\leq 6l_i} |f(t) - f(t_k)|\, d\mu_K(t) \\
&\lesssim \frac{1}{l_k^d} \int_{|t-x_k|\leq 27l_k} |f(t) - f(t_k)|\, d\mu_K(t) \\
&\leq \frac{1}{l_k^d} \int_{|t-t_k|\leq 32l_k} |f(t) - f(t_k)|\, d\mu_K(t) \\
&\lesssim \max_{|t-t_k|\leq 32l_k} |f(t) - f(t_k)|.
\end{aligned}
$$

The desired estimate follows then from the given continuity of f on (the compact set) K, by choosing l_k small enough, to which it suffices to choose a small enough $\delta > 0$.

(ii) Now we consider the case $d = n$.

Given a continuous $f \in \mathbb{B}^s_{pq}(K)$, we want to show that there exists a continuous $g \in B^{s+\frac{n-n}{p}}_{pq}(\mathbb{R}^n)$ such that $g|_K = f$. Define $f_1 : K \times \{0\} \subset \mathbb{R}^{n+1} \to \mathbb{C}$ by $f_1(x, 0) := f(x)$, obtaining in this way a continuous function in $\mathbb{B}^s_{pq}(K \times \{0\})$. Since $K \times \{0\}$ is an n-set in $\mathbb{R}^{n+1}$, with $0 < n < n + 1$, we can apply part (i) to say that there exists a continuous $g_1 \in B^{s+\frac{1}{p}}_{pq}(\mathbb{R}^{n+1})$ with $g_1|_{K\times\{0\}} = f_1$, and from here one gets that $g : \mathbb{R}^n \to \mathbb{C}$ given by $g(x) := g_1(x, 0)$ is the required function. We have taken advantage of an old trace result, which can, for example, be seen in [6, Theorem 3 in p. 19], which states that $B^s_{pq}(\mathbb{R}^n)$ can be identified with the traces on $\mathbb{R}^n \times \{0\}$ of the elements of $B^{s+\frac{1}{p}}_{pq}(\mathbb{R}^{n+1})$. □

3 Main results and proofs

With $0 \leq d \leq n$ and $0 < s \leq 1$, define

$$
H(d, s) := \begin{cases} d + 1 - s & \text{if } s < d \\ d/s & \text{if } s \geq d \end{cases}, \tag{7}
$$

or, what turns out to be the same (cf. also the end of the proof of the next proposition), $H(d, s) := \min\{d + 1 - s, d/s\}$.

Proposition 3.1. *Let $0 < s \leq 1$ and K be a d-set in $\mathbb{R}^n$, with $0 < d \leq n$. If $f \in \mathcal{C}^s(K)$ then $\dim_H \Gamma(f) \leq H(d,s)$.*

Proof. We use the inequality $\dim_H \Gamma(f) \leq \overline{\dim}_B \Gamma(f)$ and estimate the latter dimension.

As mentioned in Remark 2.9, given any $j \in \mathbb{N}$, K can be covered by $c_1 \, 2^{jd}$ cubes of side length 2^{-j} in a corresponding regular tessellation of $\mathbb{R}^n$ by dyadic cubes of sides parallel to the axes. The part of the graph of f *over* any one of such cubes can, obviously, be covered by $c_2 \, 2^{-j(s-1)}$ cubes of side length 2^{-j} of a regular tessellation of $\mathbb{R}^{n+1}$ by corresponding dyadic cubes of sides parallel to the axes. Therefore $\overline{\dim}_B \Gamma(f) \leq \limsup_{j\to\infty} \frac{\log_2(c_1 c_2) + j(d-s+1)}{j} = d + 1 - s$.

Alternatively, and using again Remark 2.9, given any $j \in \mathbb{N}$, K can be covered by $c_1 \, 2^{jd/s}$ cubes of side length $2^{-j/s}$ in a corresponding regular tessellation of $\mathbb{R}^n$ by cubes of sides parallel to the axes. The part of the graph of f *over* any one of such cubes can, obviously, be covered by c_2 parallelepipeds of height 2^{-j}, so that the whole graph can be covered by $c_1 \, c_2 \, 2^{jd/s}$ of such parallelepipeds. Since each one of these is covered by at most 2^{n+1} cubes of side length 2^{-j} of a regular tessellation of $\mathbb{R}^{n+1}$ by corresponding dyadic cubes of sides parallel to the axes, then we see that $c_3 \, 2^{jd/s}$ of the latter cubes are enough to cover $\Gamma(f)$, hence $\overline{\dim}_B \Gamma(f) \leq \limsup_{j\to\infty} \frac{\log_2(c_3) + jd/s}{j} = d/s$.

Observe now that, apart from the obvious case $s = 1$, the inequality $d/s \leq d+1-s$ holds if, and only if, $d \leq s$, which concludes the proof. $\square$

Let $[0, 2\pi]$ be endowed with its uniform Lebesgue measure, so that it becomes a probability space. In what follows, Π shall stand for the product space $([0, 2\pi]^{\mathbb{N}})^n$ of n copies of the infinite product space $[0, 2\pi]^{\mathbb{N}}$. The elements of Π shall usually be denoted by θ and we shall commit the abuse of notation of denoting by $d\theta$ both the measure in Π and integration with respect to such measure, as in $\int_\Pi d\theta$.

Theorem 3.2. *Let $\rho > 1$, $0 < s < 1$, $\theta = ((\theta_{ij})_{j\in\mathbb{N}})_{i=1,\dots,n} \in \Pi$ and $W_{s,\theta}$ be the function defined by*

$$W_{s,\theta}(x) := \sum_{i=1}^{n} \sum_{j=0}^{\infty} \rho^{-js} \cos(\rho^j x_i + \theta_{ij}), \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

Let $0 < d \leq n$ and K be a d-set. Then

$$\dim_H \Gamma(W_{s,\theta}|_K) = H(d,s) \quad \theta\text{-a.e.},$$

where $H(d,s)$ is as defined in (7).

Proof. Due to Proposition 3.1 and Example 2.16, the inequality $\leq$ is clear, even for all θ.

In order to prove the opposite inequality, we use the criteria of Remark 2.4(d).

Let μ_θ be the Borel measure supported on $\Gamma(W_{s,\theta}|_K)$ defined by $\mu_K \circ (I, W_{s,\theta})^{-1}$, where I is the identity in $\mathbb{R}^n$ and μ_K is the mass distribution supported on K according to Definition 2.7.

Given $t > 0$,

$$\begin{aligned}
&\int_\Pi \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \frac{1}{|P-Q|^t}\, d\mu_\theta(P)\, d\mu_\theta(Q)\, d\theta \\
&= \int_\Pi \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|(I, W_{s,\theta})(x) - (I, W_{s,\theta})(y)|^t}\, d\mu_K(x)\, d\mu_K(y)\, d\theta \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_\Pi \frac{1}{(|x-y|^2 + (W_{s,\theta}(x) - W_{s,\theta}(y))^2)^{t/2}}\, d\theta\, d\mu_K(x)\, d\mu_K(y) \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{1}{(|x-y|^2 + z^2)^{t/2}}\, (d\theta \circ A^{-1})(z)\, d\mu_K(x)\, d\mu_K(y),
\end{aligned} \tag{8}$$

where, for each fixed $x, y \in \mathbb{R}^n$, $A : \Pi \to \mathbb{R}$ is the function defined by $A(\theta) := W_{s,\theta}(x) - W_{s,\theta}(y)$.

Now observe that $A(\theta) = \sum_{i=1}^n A_i(\theta)$, with

$$A_i(\theta) = \sum_{j=0}^{\infty} q_{ij} \sin(r_{ij} + \theta_{ij}),$$

where q_{ij} and r_{ij} do not depend on θ. Adapting [5, pp. 797-798] to our setting, under the assumption $0 < |x_i - y_i| < \frac{\pi}{\rho^2}$, the measure $d\theta \circ A^{-1}$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}$, with density function h_i satisfying the estimate

$$h_i(z) \leq C\, |x_i - y_i|^{-s}, \tag{9}$$

where the positive constant C depends only on ρ. It is also easily seen that

$$d\theta \circ (A_i, \sum_{\substack{k=1\\k\neq i}}^n A_k)^{-1} = (d\theta \circ A_i^{-1}) \otimes (d\theta \circ (\sum_{\substack{k=1\\k\neq i}}^n A_k)^{-1}),$$

hence the density function h of $d\theta \circ A^{-1}$ is given by the convolution of the density functions h_i of $d\theta \circ A_i^{-1}$ and, say, $\breve{h}_i$ of $d\theta \circ (\sum_{\substack{k=1\\k\neq i}}^n A_k)^{-1}$. Fixing now an i such that $|x_i - y_i| = \max_{1\leq k\leq n} |x_k - y_k|$ and assuming that $0 < |x - y| < \frac{\pi}{\rho^2}$, from (9) we then get

$$h(z) = (h_i \star \breve{h}_i)(z) \leq (\sup_{w\in\mathbb{R}} h_i(w)) \int_{\mathbb{R}} \breve{h}_i(t)\, dt \leq C\, n^{s/2}\, |x-y|^{-s}, \qquad \forall z \in \mathbb{R}, \tag{10}$$

where C is the same constant as in (9).

Returning to (8), we can now write, taking into account that the hypotheses guarantee that $(\mu_K \otimes \mu_K)(\{x = y\}) = 0$,

$$
\begin{aligned}
&\int_{\Pi} \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \frac{1}{|P-Q|^t} \, d\mu_\theta(P) \, d\mu_\theta(Q) \, d\theta \qquad (11) \\
&= \int_{\substack{\mathbb{R}^n \times \mathbb{R}^n \\ |x-y| \geq \frac{\pi}{\rho^2}}} \int_{\mathbb{R}} \frac{h(z)}{(|x-y|^2 + z^2)^{t/2}} \, dz \, d(\mu_K \otimes \mu_K)(x,y) \\
&\quad + \int_{\substack{\mathbb{R}^n \times \mathbb{R}^n \\ 0 < |x-y| < \frac{\pi}{\rho^2}}} \int_{\mathbb{R}} \frac{h(z)}{(|x-y|^2 + z^2)^{t/2}} \, dz \, d(\mu_K \otimes \mu_K)(x,y),
\end{aligned}
$$

where the first term on the right-hand side is clearly finite, while, due to (10), the second term can be estimated from above by

$$
\begin{aligned}
&C\, n^{s/2} \int_{\substack{\mathbb{R}^n \times \mathbb{R}^n \\ 0 < |x-y| < \frac{\pi}{\rho^2}}} \int_{\mathbb{R}} \frac{|x-y|^{-s}}{(|x-y|^2 + z^2)^{t/2}} \, dz \, d(\mu_K \otimes \mu_K)(x,y) \\
&= C\, n^{s/2} \int_{\substack{\mathbb{R}^n \times \mathbb{R}^n \\ 0 < |x-y| < \frac{\pi}{\rho^2}}} |x-y|^{-s+1-t} \int_{\mathbb{R}} \frac{1}{(1+w^2)^{t/2}} \, dw \, d(\mu_K \otimes \mu_K)(x,y). \qquad (12)
\end{aligned}
$$

We now need to split the proof in two cases, in order to proceed.

Case $d > s$:

Consider

$$t_m := d + 1 - s - \frac{1}{m},$$

for sufficiently large $m \in \mathbb{N}$ so that $d - s > \frac{2}{m}$. Then, using t_m in the place ot t, the inner integral in (12) can be estimated from above by

$$\int_{|w| \leq 1} dw + \int_{|w| > 1} \frac{1}{|w|^{t_m}} \, dw \ \leq \ 2 + \frac{4}{d-s},$$

hence (12) can be estimated from above by

$$
\begin{aligned}
&c_1 \int_{\substack{\mathbb{R}^n \times \mathbb{R}^n \\ 0 < |x-y| < \frac{\pi}{\rho^2}}} |x-y|^{-s+1-t_m} \, d(\mu_K \otimes \mu_K)(x,y) \\
&\leq \ c_1 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{-d+1/m} \, d\mu_K(x) \, d\mu_K(y) \\
&\leq \ c_2 \int_K \int_0^{\operatorname{diam} K} r^{1/m-1} \, dr \, d\mu_K(y) \ < \ \infty,
\end{aligned}
$$

where we have used Example 2.14. Therefore, we have proved the finiteness of (11) when using $t = t_m = d + 1 - s - \frac{1}{m}$ for any sufficiently large natural m, and have shown in particular, for any such number t_m, that

$$\int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \frac{1}{|P-Q|^{t_m}} \, d\mu_\theta(P) \, d\mu_\theta(Q) < \infty \qquad \theta\text{-a.e..}$$

Consequently, by Remark 2.4(d),

$$\dim_H(\Gamma(W_{s,\theta}|_K) \geq t_m = d + 1 - s - \frac{1}{m} \qquad \theta\text{-a.e.}$$

for any m large enough. Since this is a countable number of possibilities, we can also state that, for almost all $\theta \in \Pi$, $\dim_H(\Gamma(W_{s,\theta}|_K) \geq d + 1 - s - \frac{1}{m}$ for all previously considered numbers m, so that the required result follows after letting m tend to infinity.

Case $d \leq s$:

Here we shall take advantage of the fact that the support of $d\theta \circ A^{-1}$ is contained in $[-c\,|x-y|^s, c\,|x-y|^s]$, for some positive constant c (independent of x and y). That this is the case follows from Example 2.16.

We return then to the decomposition given above for (11) and observe that we can replace the integral over $\mathbb{R}$ in the second term in that decomposition by a corresponding integral over $[-c\,|x-y|^s, c\,|x-y|^s]$, so that instead of (12) we can write, up to a constant factor,

$$\int_{\substack{\mathbb{R}^n \times \mathbb{R}^n \\ 0<|x-y|<\frac{\pi}{\rho^2}}} |x-y|^{-s+1-t} \int_0^{c\,|x-y|^{s-1}} \frac{1}{(1+w^2)^{t/2}}\, dw\, d(\mu_K \otimes \mu_K)(x,y), \tag{13}$$

where, moreover, $c\,|x-y|^{s-1}$ can, without loss of generality, be assumed to be greater than 1.

Consider now

$$t_m := \frac{d}{s} - \frac{1}{m},$$

for sufficiently large $m \in \mathbb{N}$ so that $t_m > 0$. Then, using t_m in the place ot t, the inner integral in (13) can be estimated from above by

$$\int_0^1 \frac{1}{(1+w^2)^{t_m/2}}\, dw + \int_1^{c\,|x-y|^{s-1}} \frac{1}{(1+w^2)^{t_m/2}}\, dw \;<\; \frac{c^{1-t_m}}{1-t_m}\, |x-y|^{(1-t_m)(s-1)},$$

hence (13) can be estimated from above by

$$\begin{aligned}
&\frac{c_3}{1-t_m} \int_{\substack{\mathbb{R}^n \times \mathbb{R}^n \\ 0<|x-y|<\frac{\pi}{\rho^2}}} |x-y|^{-s+1-t_m+(1-t_m)(s-1)}\, d(\mu_K \otimes \mu_K)(x,y) \\
&\leq \frac{c_3}{1-t_m} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{-d+s/m}\, d\mu_K(x)\, d\mu_K(y) \\
&\leq \frac{c_4}{1-t_m} \int_K \int_0^{\operatorname{diam} K} r^{s/m-1}\, dr\, d\mu_K(y) \;<\; \infty,
\end{aligned}$$

where we have used Example 2.14. The rest of the proof follows as in the previous case, the difference being that now $t_m = \frac{d}{s} - \frac{1}{m}$, therefore tends to $\frac{d}{s}$ when m goes to infinity. □

Theorem 3.3. *Consider $0 < d \leq n$ and K a d-set. Let $1 \leq p \leq \infty$, $0 < q \leq \infty$ and $0 < s \leq 1$. Let f be any real continuous function in $\mathbb{B}^s_{pq}(K)$. Then $\dim_H \Gamma(f) \leq H(d,s)$.*

Proof. We deal first with the case $0 < s < 1$.

We start by remarking that $\mathbb{B}^s_{pq}(K) \hookrightarrow \mathbb{B}^s_{1q}(K) \hookrightarrow \mathbb{B}^s_{1\infty}(K)$. The first of these embeddings comes from Proposition 2.25; the second one is a direct consequence of a well-known corresponding embedding for Besov spaces on $\mathbb{R}^n$. Since $H(d,s)$ does not depend on p nor q, it is then enough to prove our Theorem for the Besov spaces $\mathbb{B}^s_{1\infty}(K)$.

Given any real continuous function $f \in \mathbb{B}^s_{1\infty}(K)$, let $g \in B^{s+n-d}_{1\infty}(\mathbb{R}^n)$ be a continuous extension of f (there exists one, by Proposition 2.26). Because K is bounded, we can, without loss of generality, also assume that g is compactly supported (if necessary, we can always multiply it by a suitable cut-off function). By Definition 2.18, we can write

$$g = \sum_{j\in\mathbb{N}_0} \sum_{l=1}^{L_j} \sum_{m\in\mathbb{Z}^n} \lambda^l_{jm} 2^{-jn/2} \Psi^l_{jm}, \qquad \text{unconditional convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{14}$$

where

$$\sup_{\substack{j\in\mathbb{N}_0 \\ l=1,\ldots,L_j}} 2^{j(s-d)} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda^l_{jm}| \Big) < \infty. \tag{15}$$

Notice also, in view of Remark 2.19, that the convergence in (14) is also uniform and that, given each $j \in \mathbb{N}_0$, only a finite number of coefficients λ^l_{jm} are different from zero.

Denoting

$$h := \sum_{j\in\mathbb{N}} \sum_{l=1}^{L_j} \sum_{m\in\mathbb{Z}^n} \lambda^l_{jm} 2^{-jn/2} \Psi^l_{jm} = \sum_{j\in\mathbb{N}} \sum_{l=1}^{L} \sum_{m\in\mathbb{Z}^n} \lambda^l_{jm} 2^{-n/2} \psi^l(2^{j-1}\cdot - m)$$

(which, clearly, is also a continuous function belonging to $B^{s+n-d}_{1\infty}(\mathbb{R}^n)$), we can write

$$f = \sum_{m\in\mathbb{Z}^n} \lambda^l_{0m} \Psi^l_{0m}|_K + h|_K.$$

Since the sum on m is a Lipschitz function (as we have remarked above, this sum actually has only a finite number of non-zero terms), then $\Gamma(f)$ and $\Gamma(h|_K)$ have the same Hausdorff dimension (this follows from Remark 2.4(c)). Therefore, our proof will be finished if we show that $\dim_H \Gamma(h|_K) \leq H(d,s)$. This is what we are going to prove next, assuming, for ease of writing, that our h is simply given by

$$h := \sum_{j\in\mathbb{N}} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} \psi(2^{j-1}\cdot - m) = \sum_{j\in\mathbb{N}} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} \psi_{jm}. \tag{16}$$

So, we got rid of the finite summation in l and of the unimportant factor $2^{-n/2}$ and simplified the notation for the λ's and ψ's (introducing also the simplification $\psi_{jm} := \psi(2^{j-1} \cdot -m)$). In this way we keep the essential features of the method without unecessarily overcrowding the notation. If we were to consider the exact form of h in what follows, after some point we could indeed get rid of the finite summation in l without changing the estimates that are obtained up to multiplicative positive constants.

In order to estimate $\dim_H \Gamma(h|_K)$ from above, we are going to estimate, also from above, the quantities $\mathcal{H}^t_{\sqrt{n+1}\,2^{-j_1+1}}(\Gamma(h|_K))$, for $t \geq 0$ and $j_1 \in \mathbb{N} \setminus \{1\}$.

We start by estimating

$$\mathcal{H}^t_{\sqrt{n+1}\,2^{-j_1+1}}(\Gamma(h_0|_K + \sum_{j=j_1}^{j_2-1} h_j|_K),$$

where $j_2 > j_1$ (with $j_2 \in \mathbb{N}$),

$$h_0 := \sum_{j=1}^{j_1-1} \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\psi_{jm} \quad \text{and} \quad h_j := \sum_{m \in \mathbb{Z}^n} \lambda_{jm}\psi_{jm}, \quad j = j_1, \ldots, j_2 - 1,$$

by applying Lemma 2.11 a finite number of times. In what follows we use the notations Q_j and $\mathcal{Q}_j$ with the same meaning as in that lemma (assuming further that the coverings $\mathcal{Q}_j$ are minimal) and each time we start from a covering of

$$\Gamma(h_0|_K + \sum_{j=j_1}^{k} h_j|_K), \quad k = j_1 - 1, j_1, \ldots, j_2 - 2$$

(with the understanding that when $k = j_1 - 1$ we are starting from a covering of $\Gamma(h_0|_K))$ by $(n+1)$-cubes of side length at least 2^{-k} and such that *over* each Q_k each point between the *levels* $m_{Q_k} := \inf_{Q_k \cap K}(h_0|_K + \sum_{j=j_1}^{k} h_j|_K)$ and $M_{Q_k} := \sup_{Q_k \cap K}(h_0|_K + \sum_{j=j_1}^{k} h_j|_K)$ belongs to one of those $(n+1)$-cubes. Therefore, by applying Lemma 2.11, each time we conclude that the number of $(n+1)$-cubes of side length $2^{-(k+1)}$ that one needs to add to the previous covering, in order to get a covering of $\Gamma(h_0|_K + \sum_{j=j_1}^{k+1} h_j|_K)$ by $(n+1)$-cubes of side length at least $2^{-(k+1)}$, and such that *over* each Q_{k+1} each point between the *levels* $m_{Q_{k+1}}$ and M_{Q_l} belongs to one of those $(n+1)$-cubes, is bounded above by

$$\sum_{Q_{k+1} \in \mathcal{Q}_{k+1}} (2^{k+2} \sup_{y \in Q_{k+1}} |h_{k+1}|_K(y)| + 2).$$

Hence, starting from a covering of $\Gamma(h_0|_K)$ by $(n+1)$-cubes of side length $2^{-(j_1-1)}$ built, with the help of the concept of oscillation, *over* each $Q_{j_1-1} \in \mathcal{Q}_{j_1-1}$, whose

number is bounded above by

$$\sum_{Q_{j_1-1}\in\mathcal{Q}_{j_1-1}} (2^{j_1-1}\mathrm{osc}_{Q_{j_1-1}} h_0|_K + 2),$$

and applying Lemma 2.11 repeatedly (a total number of $j_2 - j_1$ times), we get the following estimates (the constants might depend on t), where we have also used Remark 2.9 to estimate the number of elements of each $\mathcal{Q}_j$:

$$\begin{aligned}
&\mathcal{H}^t_{\sqrt{n+1}\,2^{-j_1+1}}(\Gamma(h_0|_K + \sum_{j=j_1}^{j_2-1} h_j|_K)) \\
&\lesssim\ 2^{-j_1 t} \sum_{Q_{j_1-1}\in\mathcal{Q}_{j_1-1}} (2^{j_1-1}\mathrm{osc}_{Q_{j_1-1}} h_0|_K + 2) \\
&\quad + \sum_{j=j_1}^{j_2-1} 2^{-jt} \sum_{Q_j\in\mathcal{Q}_j} (2^{j+1}\sup_{y\in Q_j} |h_j|_K(y)| + 2) \\
&\lesssim\ 2^{-j_1(t-d)} + 2^{-j_1(t-1)} \sum_{Q_{j_1-1}\in\mathcal{Q}_{j_1-1}} \mathrm{osc}_{Q_{j_1-1}} h_0|_K \qquad (17) \\
&\quad + \sum_{j=j_1}^{j_2-1} 2^{-j(t-d)} + \sum_{j=j_1}^{j_2-1} 2^{-j(t-1)} \sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|.
\end{aligned}$$

The estimate in the last term is possible because of the controlled overlapping between the support of each ψ_{jm} and the different Q_j's (i.e., each $\mathrm{supp}\,\psi_{jm}$ intersects only a finite number of Q_j's and this number can be bounded above by a constant independent of j).

Now, if one wants to estimate $\mathcal{H}^t_{\sqrt{n+1}\,2^{-j_1+1}}(\Gamma(h|_K))$ instead, notice that the Lemma 2.11 can again be used, and we just have to find out what is the contribution, coming from $\sum_{j=j_2}^{\infty} h_j|_K$, that we need to add to the right-hand side of (17). Actually, this could have been done at the same time we added the contribution of the term $h_{j_1}|_K$, resulting in the extra term

$$\begin{aligned}
&2^{-j_1(t-1)} \sum_{Q_{j_1}\in\mathcal{Q}_{j_1}} \sup_{y\in Q_{j_1}} |\sum_{j=j_2}^{\infty} h_j|_K(y)| \\
&\lesssim\ 2^{-j_1(t-1)} 2^{j_1 d} \sup_{y\in\mathbb{R}^n} |\sum_{j=j_2}^{\infty} h_j|_K(y)| \ =:\ 2^{-j_1(t-1)} 2^{j_1 d} C_{j_2}.
\end{aligned}$$

However, by the already mentioned uniform convergence of the sum defining h, we have that C_{j_2} tends to 0 as j_2 goes to infinity. Therefore, by choosing j_2 large enough (in dependence of j_1) so that $C_{j_2} 2^{j_1} \leq 1$, the last contribution is just of the type $2^{-j_1(t-d)}$. This is the same as the first term in (17), and both can be absorbed by

the *third* term in that expression. Hence, for such a choice of j_2,

$$\begin{aligned}&\mathcal{H}^t_{\sqrt{n+1}\,2^{-j_1+1}}(\Gamma(h|_K)) \qquad (18)\\ &\lesssim \sum_{j=j_1}^{j_2-1} 2^{-j(t-d)} + 2^{-j_1(t-1)} \sum_{Q_{j_1-1}\in\mathcal{Q}_{j_1-1}} \operatorname{osc}_{Q_{j_1-1}} h_0|_K + \sum_{j=j_1}^{j_2-1} 2^{-j(t-1)} \sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|.\end{aligned}$$

Now we estimate separately each one of the three distinguished terms above (in all cases ignoring unimportant multiplicative constants):

The first one is dominated by $2^{-j_1(t-d)}$ under the assumption $t > d$.

The second one is dominated by

$$\begin{aligned}&2^{-j_1(t-1)} \sum_{Q_{j_1-1}\in\mathcal{Q}_{j_1-1}} \sum_{j=1}^{j_1-1} \sum_m \operatorname{osc}_{Q_{j_1-1}}(\lambda_{jm}\psi_{jm}|_K)\\ &\leq\ 2^{-j_1(t-1)} \sum_{j=1}^{j_1-1} \sum_m \sum_{Q_{j_1-1}\in\mathcal{Q}_{j_1-1}} |\lambda_{jm}|\, |\nabla\psi_{jm}(\xi_{j_1})|\, 2^{-j_1}, \qquad (19)\end{aligned}$$

where ξ_{j_1} is chosen in Q_{j_1-1} in accordance with the mean value theorem (which we have just used above) and in $\sum_m$ the m is restricted to the values for which $\operatorname{supp}\psi_{jm}$ intersects K. Such a number of m's can clearly be estimated from above by 2^{jd} (cf. Remark 2.9 and Definition 2.17). Next we remark that $|\nabla\psi_{jm}(\xi_{j_1})| \lesssim 2^j$ and that in the inner sum in (19) we only need to consider the Q_{j_1-1}'s which intersect $\operatorname{supp}\psi_{jm}$. It is not difficult to see that such a number of Q_{j_1-1}'s can be estimated from above by $2^{(j_1-j)d}$. Putting all this together, and using also the estimate (15) and the hypothesis $0 < s < 1$, the second term on the right-hand side of (18) is dominated by

$$\begin{aligned}&2^{-j_1(t-1)} \sum_{j=1}^{j_1-1} 2^{j-j_1} 2^{(j_1-j)d} \sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|\\ &\lesssim\ 2^{-j_1(t-d)} j_1 \max_{1\leq j\leq j_1-1} 2^{-j(s-1)} \ \approx\ j_1 2^{-j_1(t-d+s-1)}.\end{aligned}$$

Finally, again by using the estimate (15), the third term on the right-hand side of (18) can be dominated by $2^{-j_1(t-d+s-1)}$ under the assumption $t > d-s+1$.

Altogether, and under the assumption $t > d-s+1$ (which, in particular, implies that $t > d$, due to the hypothesis $0 < s < 1$), we have obtained that

$$\mathcal{H}^t_{\sqrt{n+1}\,2^{-j_1+1}}(\Gamma(h|_K)) \lesssim j_1 2^{-j_1(t-d+s-1)},$$

from which it follows that

$$0 \leq \mathcal{H}^t(\Gamma(h|_K)) = \lim_{\delta\to 0+} \mathcal{H}^t_\delta(\Gamma(h|_K)) = \lim_{j_1\to\infty} \mathcal{H}^t_{\sqrt{n+1}\,2^{-j_1+1}}(\Gamma(h|_K)) \leq 0,$$

that is, $\mathcal{H}^t(\Gamma(h|_K)) = 0$.

This being true for any $t > d+1-s$, we obtain, by definition, that $\dim_H(\Gamma(h|_K)) \leq d+1-s$, that is, $\dim_H(\Gamma(h|_K)) \leq H(d,s)$ in the case $s \leq d$.

We assume now that $s > d$ and show that also $\dim_H(\Gamma(h|_K)) \leq H(d,s)$, which means $\dim_H(\Gamma(h|_K)) \leq d/s$ in this case (cf. definition of $H(d,s)$ in (7)). As we shall see, it will be enough to estimate $\overline{\dim}_B(\Gamma(h|_K))$ and use the relation $\dim_H(\Gamma(h|_K)) \leq \overline{\dim}_B(\Gamma(h|_K))$.

Given $\nu \in \mathbb{N}$, we start by covering K by $\approx 2^{\nu d/s}$ n-cubes I_ν of side length $2^{-\nu/s}$ taken from a given regular tessellation of $\mathbb{R}^n$ by cubes of sides parallel to the axes and of such side length (we know, from Remark 2.9, that this is possible). Using the *representation* (16) for h, we can write

$$\begin{aligned}
\sum_{I_\nu} \mathrm{osc}_{I_\nu} h|_K &\leq \sum_{I_\nu} \sum_{j=1}^{\infty} \sum_{m\in\mathbb{Z}^n} \mathrm{osc}_{I_\nu}(\lambda_{jm}\psi_{jm}|_K) \\
&= \sum_{j=1}^{j_\nu-1} \sum_{m\in\mathbb{Z}^n} \sum_{I_\nu} \mathrm{osc}_{I_\nu}(\lambda_{jm}\psi_{jm}|_K) + \sum_{j=j_\nu}^{\infty} \sum_{m\in\mathbb{Z}^n} \sum_{I_\nu} \mathrm{osc}_{I_\nu}(\lambda_{jm}\psi_{jm}|_K) \\
&=: (I) + (II),
\end{aligned}$$

where $j_\nu \in \mathbb{N}$ was chosen in such a way that $\nu/s \leq j_\nu \leq \nu/s + 1$. In particular, $2^{-j_\nu} \approx 2^{-\nu/s}$. Reasoning now as was done to control the second term on the right-hand side of (18), we can dominate (I) by

$$\begin{aligned}
&\sum_{j=1}^{j_\nu-1} \sum_{m} \sum_{I_\nu} |\lambda_{jm}|\, |\nabla\psi_{jm}(\xi_\nu)|\, 2^{-j_\nu} \\
&\lesssim \sum_{j=1}^{j_\nu-1} 2^{j-j_\nu} 2^{(j_\nu-j)d} \sum_{m\in\mathbb{Z}^n} |\lambda_{jm}| \lesssim j_\nu 2^{-j_\nu(s-d)} \approx \nu 2^{-\nu(s-d)/s}.
\end{aligned}$$

On the other hand, (II) can be dominated by

$$\begin{aligned}
&\sum_{j=j_\nu}^{\infty} \sum_{m\in\mathbb{Z}^n} \sum_{I_\nu} \sup_{I_\nu} |\lambda_{jm}\psi_{jm}|_K| \\
&\lesssim \sum_{j=j_\nu}^{\infty} \sum_{m\in\mathbb{Z}^n} \sup_{\mathbb{R}^n} |\lambda_{jm}\psi_{jm}|_K| \lesssim \sum_{j=j_\nu}^{\infty} 2^{-j(s-d)} \approx 2^{-\nu(s-d)/s}.
\end{aligned}$$

The first estimate above is possible because of the controlled overlapping between the support of each ψ_{jm} and the different I_ν's (i.e., each $\mathrm{supp}\, \psi_{jm}$ intersects only a finite number of I_ν's and this number can be bounded above by a constant independent of j and j_ν, due to the fact that here we have $j \geq j_\nu$). The second estimate follows

from (15). Summing up,

$$\sum_{I_\nu} \mathrm{osc}_{I_\nu} h|_K \lesssim \nu 2^{-\nu(s-d)/s}.$$

Now consider a covering of the graph of $h|_K$ by $(n+1)$-cubes of side length $2^{-\nu}$ taken from a corresponding regular tessellation of $\mathbb{R}^{n+1}$ by dyadic cubes of sides parallel to the axes. Recalling the assumption $0 < s < 1$, it is clear that the number of such cubes does not exceed $\sum_{I_\nu}(2^\nu \mathrm{osc}_{I_\nu} h|_K + 1)$. Then, from the estimate above we see that this number is dominated by

$$\sum_{I_\nu} 1 + 2^\nu \sum_{I_\nu} \mathrm{osc}_{I_\nu} h|_K \;\lesssim\; 2^{\nu d/s} + 2^\nu \nu 2^{-\nu(s-d)/s} \;\approx\; \nu 2^{\nu d/s}.$$

Therefore,

$$\overline{\dim}_B(\Gamma(h|_K)) \le \lim_{\nu\to\infty} \left(\frac{\log_2 \nu}{\nu} + \frac{\nu d/s}{\nu} \right) = \frac{d}{s}.$$

We recall that we have been assuming $0 < s < 1$. We deal now with the case $s = 1$.

Given a real continuous function f in $\mathbb{B}^1_{pq}(K)$, then we also have $f \in \mathbb{B}^s_{pq}(K)$ for any $s \in (0,1)$, therefore $\dim_H \Gamma(f) \le H(d,s)$ for any such s. Hence, if $d \ge 1$, also $d > s$ and $\dim_H \Gamma(f) \le d + 1 - s$; letting $s \to 1^-$, we get $\dim_H \Gamma(f) \le d = H(d,1)$. If $d < 1$, choose any $s \in [d,1)$, so that $s \ge d$ and, therefore, $\dim_H \Gamma(f) \le d/s$; again letting $s \to 1^-$, it follows $\dim_H \Gamma(f) \le d = H(d,1)$. $\square$

Corollary 3.4. *Consider $0 < d \le n$ and K a d-set. Let $1 \le p \le \infty$, $0 < q \le \infty$ and $0 < s \le 1$. Then the estimates of Proposition 3.1 and Theorem 3.3 are sharp. That is, $\sup_f \dim_H \Gamma(f) = H(d,s)$, where the supremum is taken over all real continuous functions belonging either to $\mathcal{C}^s(K)$ or to $\mathbb{B}^s_{pq}(K)$.*

Proof. Consider first the case of the spaces $\mathcal{C}^s(K)$. If $0 < s < 1$, it follows from Example 2.16 and Theorem 3.2 that there exists a real continuous function in $\mathcal{C}^s(K)$ — namely $W_{s,\theta}|_K$, for some θ — the graph of which has Hausdorff dimension exactly equal to $H(d,s)$. If $s = 1$, then $H(d,s) = H(d,1) = d$ and the result follows from Lemma 2.10.

As to the spaces $\mathbb{B}^s_{pq}(K)$, for $s = 1$ it follows exactly as just pointed out, so we only need to consider $s \in (0,1)$:

Let $\varepsilon \in (0, 1-s)$, so that $0 < s+\varepsilon < 1$. Recall — see Example 2.16 — that $W_{s+\varepsilon,\theta}$ is Hölder continuous of exponent $s+\varepsilon$ on any bounded subset of $\mathbb{R}^n$. Considering then an open bounded set $V \supset K$ and a function $\psi \in \mathcal{C}^1(\mathbb{R}^n)$ with $\psi \equiv 1$ on K and $\psi \equiv 0$ outside V, we have $\psi W_{s+\varepsilon,\theta} \in \mathcal{C}^{s+\varepsilon}(\mathbb{R}^n)$. Since the latter space is contained in

$B^{s+\varepsilon}_{\infty\infty}(\mathbb{R}^n)$ — cf. [12, pp. 4, 5 and 17] —, and this, in turn, is embedded in $B^s_{\infty q}(\mathbb{R}^n)$, then, using Remark 2.23, Definition 2.24 and Proposition 2.25, we can write

$$f_\varepsilon := W_{s+\varepsilon,\theta}|_K = (\psi W_{s+\varepsilon,\theta})|_K = tr_K(\psi W_{s+\varepsilon,\theta}) \in \mathbb{B}^s_{\infty q}(K) \subset \mathbb{B}^s_{pq}(K).$$

Therefore, given any $\varepsilon \in (0, 1-s)$ there exists a real continuous function $f_\varepsilon \in \mathbb{B}^s_{pq}(K)$ the graph of which has Hausdorff dimension equal to $H(d, s+\varepsilon)$. If $s < d$, restrict further the ε to be also in $(0, d-s)$, so that $H(d, s+\varepsilon) = d+1-s-\varepsilon$, hence $\sup_{f_\varepsilon} \dim_H \Gamma(f) = d+1-s = H(d,s)$. If $s \geq d$, then also $s+\varepsilon \geq d$, so that $H(d, s+\varepsilon) = d/(s+\varepsilon)$, hence $\sup_{f_\varepsilon} \dim_H \Gamma(f) = d/s = H(d,s)$ too. □

References

[1] M. Bricchi. *Tailored function spaces and related h-sets*. PhD thesis, Friedrich-Schiller-Universität Jena, 2001.

[2] A. Carvalho. Box dimension, oscillation and smoothness in function spaces. *J. Funct. Spaces Appl.*, 3(3):287–320, 2005.

[3] A. Deliu and B. Jawerth. Geometrical dimension versus smoothness. *Constr. Approx.*, 8:211–222, 1992.

[4] K. J. Falconer. *Fractal Geometry*. John Wiley & Sons, Chichester, 1990.

[5] B. Hunt. The Hausdorff dimension of graphs of Weierstrass functions. *Proc. Amer. Math. Soc.*, 126:791–800, 1998.

[6] A. Jonsson and H. Wallin. *Function Spaces on Subsets of* $\mathbb{R}^n$, volume 2 of *Math. Reports*. Harwood Acad. Publ., 1984.

[7] J.-P. Kahane. *Some random series of functions*. Cambridge Univ. Press, 2nd edition, 1985.

[8] A. Kamont and B. Wolnik. Wavelet Expansions and Fractal Dimensions. *Constructive Approximation*, 15(1):97–108, 1999.

[9] S. Moura. Function spaces of generalised smoothness. *Dissertationes Math.*, 398:88 pp., 2001.

[10] S. Moura. *Function Spaces of Generalised Smoothness, Entropy Numbers, Applications*. PhD thesis, University of Coimbra, 2001.

[11] F. Roueff. *Dimension de Hausdorff du graphe d'une fonction continue: une étude analytiqueet statistique*. PhD thesis, Ecole Nat. Supér. Télécom., 2000.

[12] H. Triebel. *Theory of Function Spaces II.* Birkhäuser, Basel, 1992.

[13] H. Triebel. *Fractals and Spectra.* Birkhäuser, Basel, 1997.

[14] H. Triebel. Fractal analysis, an approach via function spaces. *Jahresbericht DMV*, 104(4):171–199, 2002. English translation.

[15] H. Triebel. *Theory of Function Spaces III.* Birkhäuser, Basel, 2006.